Webster's equation in acoustic wave guides

An improved version of the "Oberwolfach lecture" in November 2015

Jarmo Malinen

Aalto University
Dept. of Mathematics and Systems Analysis, School of Science
Dept. of Signal Processing and Acoustics, School of Electrical Engineering

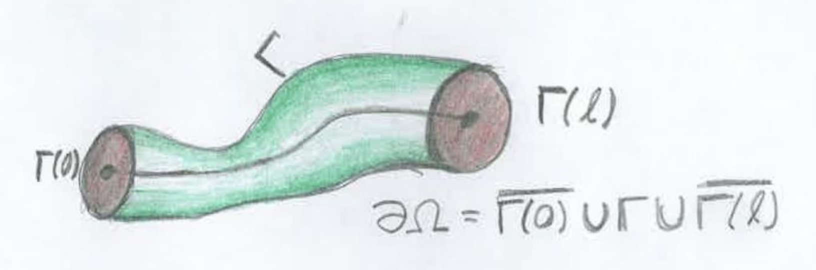
Matematiikan päivät, Turku
January 7-8, 2016

Webster's model

Waveguides by the wave equation (1)

Let $\Omega \subset \mathbb{R}^3$ be a variable diameter tubular domain whose acoustics should be modelled.

The ends of the tube are denoted by $\Gamma(0)$, $\Gamma(\ell)$, and the tube wall is Γ. Thus $\partial\Omega = \overline{\Gamma(0)} \cup \Gamma \cup \overline{\Gamma(\ell)}$.



We assume that $\Gamma(0)$ is used for active control, and the observations are done on $\Gamma(\ell)$. On Γ, some time-independent boundary condition is used.

Standing assumption: Ω is a tube of finite, non-zero intersectional diameter.

Waveguides by the wave equation (1)

The equations for the velocity potential $\phi = \phi(\mathbf{r}, t)$:

$$\begin{cases} \phi_{tt} = c^2 \Delta \phi & \text{inside the waveguide volume } \Omega, \\ \frac{\partial \phi}{\partial \nu}(\mathbf{r}, t) = 0 & \text{on the hard waveguide walls } \mathbf{r} \in \Gamma, \\ \frac{\partial \phi}{\partial \nu}(\mathbf{r}, t) = V_1(\mathbf{r}, t) & \text{on the control surface } \mathbf{r} \in \Gamma(0), \\ \rho \phi_t(\mathbf{r}, t) + R_L A(\ell) \frac{\partial \phi}{\partial \nu}(\mathbf{r}, t) = 0 & \text{on the termination surface } \mathbf{r} \in \Gamma(\ell) \end{cases}$$

The model parameters:

c	speed of sound
ρ	density of air
ν	exterior normal on $\partial\Omega$
$A(0)$, $A(\ell)$	areas of $\Gamma(0)$, $\Gamma(\ell)$
R_L	load resistance at $\Gamma(\ell)$.

Waveguides by the wave equation (2)

The point in velocity potential ϕ is that the complementary *acoustic state variables* P and V can be defined in terms of it. More precisely,

$$P(\mathbf{r}, t) := \rho\phi_t(\mathbf{r}, t) \text{ and } V(\mathbf{r}, t) := \frac{\partial\phi}{\partial\nu}(\mathbf{r}, t)$$

where P is the sound pressure (i.e., the acoustic voltage) and V is the perturbation velocity (i.e., the acoustic current per unit area).

Written in terms of the velocity potential, linear acoustics is the simplest of all classical field theories.

Cheaper model for tubular domains?

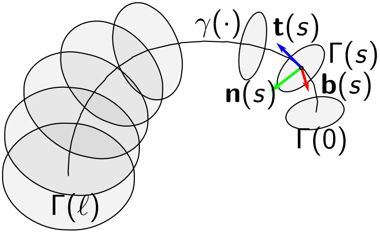
We define the slicing of the tube $\Omega \subset \mathbb{R}^3$:

$\gamma(\cdot)$ centreline of Ω, parameterised by its arc length s

ℓ length of $\gamma(\cdot)$

$\Gamma(s)$ slice of Ω, normal to $\gamma(\cdot)$ at s

$A(s)$ area of $\Gamma(s)$



Now, is there an approximate equation for the averages

$$\bar{\phi}(s,t) := \frac{1}{A(s)} \int_{\Gamma(s)} \phi dA \quad \text{for} \quad s \in [0,\ell]$$

of the velocity potential ϕ given by the wave equation on Ω?

Webster's resonator (1)

Equations for the Webster's velocity potential $\psi = \psi(s,t)$ using the intersection areas $A = A(s)$ for $s \in [0,\ell]$:

$$\begin{cases} \psi_{tt} = \frac{c^2}{A(s)} \frac{\partial}{\partial s} \left(A(s) \frac{\partial \psi}{\partial s} \right) & \text{in the waveguide } s \in [0,\ell] \\ -A(0)\psi_s(0,t) = i_1(t) & \text{at the control end } s = 0 \\ \rho \psi_t(\ell,t) + R_L A(\ell) \psi_s(\ell,t) = 0 & \text{at the termination end } s = \ell. \end{cases}$$

This is an impedance passive **strong** boundary node if the pressure output is defined as

$$p_2(t) = \rho \psi_t(\ell,t).$$

If the termination resistance $R_L = \infty$, i.e., $\psi_s(\ell,t) = 0$, then the node is even impedance conservative.

Webster's resonator (2)

It can be shown that if the non-longitudinal acoustic modes on Ω are not "strongly excited", then the solutions of the wave equation and Webster's equation safisfy

$$\bar{\phi}(s,t) \approx \psi(s,t) \quad \text{for all } s \in [0,\ell] \text{ and } t \in [0,T]$$

for not too large time intervals $[0,T]$.

Showing this in a mathematically rigorous way amounts to writing an *a posteriori* estimate (not discussed here).

Webster's equation is not only numerically lighter than the 3D wave equation model. Some problems in Webster's setting are mathematically tractable whereas the same cannot be said about the 3D wave equation.

Webster's resonator (3)

For Webster's equation, we define the acoustic state variables by

$$p(s,t) := \rho\psi_t(s,t) \quad \text{and} \quad i(s,t) := A(s)\phi_s(s,t).$$

Again, the sound pressure (i.e., the acoustic voltage) is denoted by p, and the perturbation volume velocity (i.e., the acoustic current) is denoted by i.

We need also the notion of the characteristic acoustic impedance given by

$$Z_0(s) := \frac{\rho c}{A(s)} \text{ for } s \in [0,\ell].$$

So, acoustic waveguides...

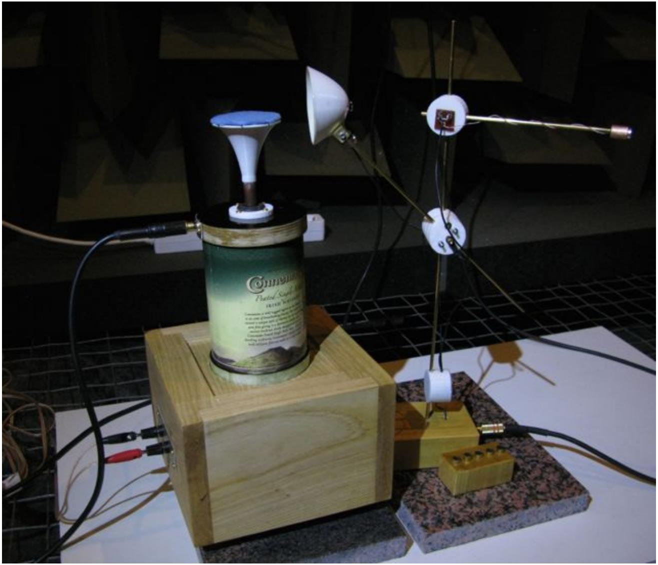
...but why?

So, acoustic waveguides...

...but why?

Let me show you some acoustic waveguides.

Acoustic glottal source (1)

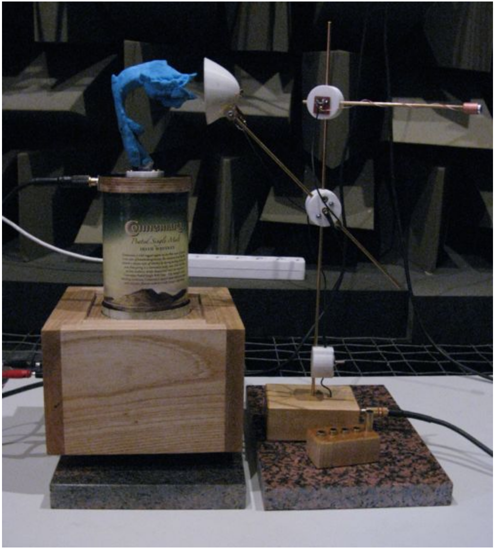


Acoustic glottal source (2)

What do we have here?

- Impedance matching by tractrix (i.e., pseudosphere) horn.
- Helmholtz-based acoustic design by FEM.
- Fast prototyping, i.e., 3D printing of complicated shapes.
- Numerical precompensation of nonidealities, based on frequency response measurements.

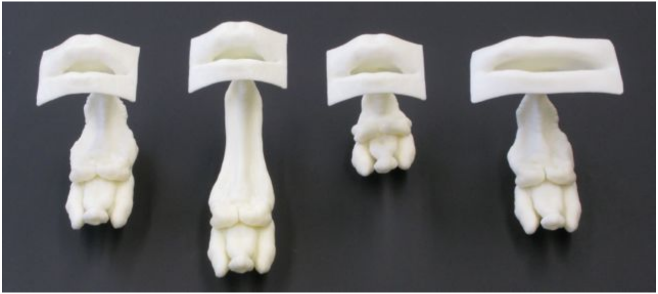
Measurement setup (1)



Measurement setup (2)

(Joint ongoing work with A. Hannukainen, J. Kuortti, and A. Ojalammi.)

Objets d'art with modified geometries



http://sinne.proartibus.fi/fi/event/evocal/

(Back to mathematics.)

Transmission impedance has no zeroes

Transmission impedance has no zeroes

The usual all-pole modelling used, for example, in Glottal Inverse Filtering algorithms is quite OK.

Acoustic waveguide as a two-port

Let us write the system described by Webster's model using electrical circuit notation.

Here p_1, p_2 are "voltages" and i_1, i_2 are "currents" at the ends of the waveguide. They are given by

$$
\begin{aligned}
p_1 &= \rho\psi_t(0,\cdot), & p_2 &= \rho\psi_t(\ell,\cdot), \\
i_1 &= A(0)\psi_s(0,\cdot), & i_2 &= A(\ell)\psi_s(\ell,\cdot)
\end{aligned}
$$

in terms of the velocity potential.

Impedance t.f. of the two-port (1)

In Laplace transform domain, we have

$$\begin{bmatrix} \hat{p}_1(\xi) \\ \hat{p}_2(\xi) \end{bmatrix} = \begin{bmatrix} Z_1(\xi) & Z_{back}(\xi) \\ Z_{tr}(\xi) & Z_2(\xi) \end{bmatrix} \begin{bmatrix} \hat{i}_1(\xi) \\ \hat{i}_2(\xi) \end{bmatrix} \quad \text{for } \xi \in \mathbb{C}^+$$

where the impedance t.f. $Z(\xi)$ is a 2×2 matrix-valued analytic function whose domain includes $\mathbb{C}^+ := \{\mathrm{Re}\, \xi > 0\}$.
We have

1. $Z(\xi) + Z(\xi)^* \geq 0$ for $\xi \in \overline{\mathbb{C}^+}$ by passivity.
2. $Z(\xi) + Z(\xi)^* > 0$ for $\xi \in \mathbb{C}^+$ by the maximum modulus theorem.

The same holds for $Z_{1,1}$, $Z_{2,2}$ in place of Z as well.

We conclude that none of the functions $Z_{1,1}$, $Z_{2,2}$, Z may have a zero in $\mathbb{C}^+$. Actually, zeroes in $i\mathbb{R}$ can be excluded as well.

The transmission impedance (1)

That $Z_{1,1}$, $Z_{2,2}$, and Z are zero-free depends only on the passivity, the compact resolvent property of the semigroup, and approximate controllability of the "admittance system".

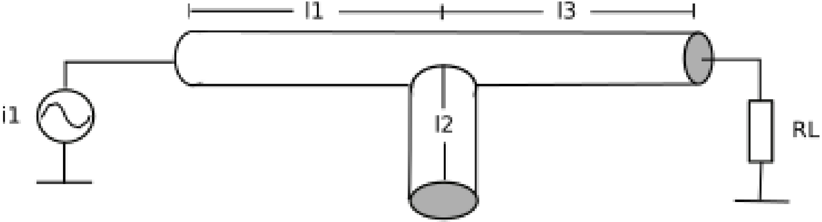
Therefore, the conclusion holds also for *transmission graphs* consisting of a finite number of Webster's resonators, coupled by Kirchhoff laws.

However, we are more interested in the the *transmission impedances* Z_{back} and Z_{tr}. Are these zero-free (at least in $\mathbb{C}^+$)?

Let us concentrate on Z_{tr}. Here $\hat{p}_2(\xi) = Z_{tr}(\xi)\hat{i}_1(\xi)$.

The transmission impedance (2)

At least, the zero-free property of Z_{tr} is not a consequence of passivity (etc.) as can be seen by considering the band stop filter realised by a transmission graph with "T" -topology:



At the wavelength $\lambda = 4\ell_2$, there is a destructive interference leading to zero impedance in parallel with R_L and the segment of length ℓ_3.

This decouples the load R_L from the current source i_1 – hence, there is a zero in transmission impedance t.f. from i_1 to R_L.

The transmission impedance (3)

For constant diameter waveguides, Webster's equation can be solved explicitly. We get

$$Z_{tr}(\xi) = \frac{Z_0 R_L}{Z_0 \cosh \frac{\xi \ell}{c} + R_L \sinh \frac{\xi \ell}{c}}$$

where $Z_0 = \rho c / A$.
At perfect impedance matching, i.e., $R_L = Z_0$, we get

$$Z_{tr}(\xi) = Z_0 e^{-\frac{\xi \ell}{c}}$$

which is the pure delay of length $\tau = \ell / c$ in the time domain.

In these cases, the transmission impedance has no zeroes in $\mathbb{C}^+$. Is this a generic property of waveguides with non-constant area functions?

Eigenvalue problem for the zeros (1)

The zeroes of Z_{tr} are eigenvalues λ of the flow inverted "admittance system" that is impedance passive as well:

$$\begin{cases} \left(\frac{\lambda}{c}\right)^2 \psi_\lambda = \frac{1}{A(s)} \frac{\partial}{\partial s}\left(A(s)\frac{\partial \psi_\lambda}{\partial s}\right) & \text{in waveguide } s \in [0, \ell], \\ \psi_\lambda(\ell, t) = (\psi_\lambda)_s(\ell, t) = 0 & \text{at the termination end } s = \ell. \end{cases}$$

If $A(s)$ is smooth in a neighbourhood of $s = \ell$, then so is ψ_λ. Then we have

$$\frac{\partial^k \psi_\lambda}{\partial s^k}(\ell) = 0 \text{ for all } k \geq 0.$$

Eigenvalue problem for the zeros (2)

We have now shown:

Theorem

There are no non-zero solutions ψ_λ, real-analytic in a neighbourhood of $s = \ell$, of the eigenvalue problem related to the "admittance system" for any λ^2.

Now, if the area function $A(\cdot)$ is real-analytic in a neighbourhood of $s = \ell$, then so is (the real or the imaginary part of) the solution ψ_λ by Holmgren's uniqueness theorem.

Neither can a solution $\psi(s, t) = e^{\lambda t}\psi_\lambda(s)$ of Webster's equation vanish in an open set $s \in I$ for all $t \geq 0$ unless $\psi(s, t) \equiv 0$.

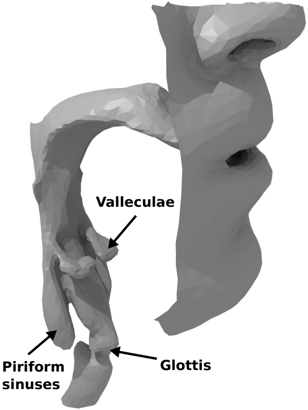
We conclude that Z_{tr} has no zeros in $\mathbb{C}$ as claimed. H. Zwart has shown recently by semigroup techniques that Z_{tr} has no zeroes in $\mathbb{C}_+$ which is good enough as well.

An application: Glottal inverse filtering

Reconstruction of the volume velocity signal $i(t)$ at the vocal folds position, based on measurements of the sound pressure $p(t)$ at mouth?

It is possible to estimate the poles of Z_{tr} from $\hat{p}$. The usual all-pole (zero-free) modelling of Z_{tr} is consistent with Webster's model!

There is *a priori* knowledge of i as well that helps the separation of i and Z_{tr} contributions in p.



(Joint ongoing work with P. Alku and H. Zwart.)

Sensitivity of spectrum

Webster's eigenvalue problem (1)

We are interested in the resonances of Webster's model, given by

$$\lambda^2\psi_\lambda = \frac{c^2}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial\psi_\lambda}{\partial s}\right) \qquad \text{for } s \in [0,\ell].$$

(To simplify matters, we assume that the boundary conditions are such that $\lambda^2 \in \mathbb{R}$.)

There is typically an infinite sequence of solutions $(\lambda, \psi_\lambda) \in \overline{\mathbb{C}^-} \times H^1(0,1) \setminus \{0\}$ that satisfy $\psi_\lambda \in C^\infty[0,1]$ if $A(\cdot)$ is smooth. If λ is an eigenvalue, then so is $\overline{\lambda}$.

So we have two eigenvalue sequences $\{\lambda_j^+\}$ and $\{\lambda_j^-\}$ with

$$0 \leq Im\,\lambda_1^+ \leq Im\,\lambda_2^+ \leq \ldots$$

and $\lambda_j^- = \overline{\lambda_j^+}$.

Resonant frequencies are given by $R_j = Im\,\lambda_j^+/2\pi$, $j = 1, 2, \ldots$.

Spectral perturbation (1)

Suppose that the area function $A(\cdot)$ and another (not necessarily positive) function $B(\cdot)$ are given, both defined on $[0, \ell]$.

Given a fixed solution $\left(\lambda_j, \psi_{\lambda_j}\right)$, $j \in \mathbb{N}$, of the resonance problem with $A(\cdot)$, approximate the perturbed solution

$$\left(\lambda_j(\epsilon), \psi_{\lambda_j}(\epsilon)\right) \in \mathbb{C}^- \times H^1(0,1) \setminus \{0\}$$

of Webster's model, corresponding to area function

$$A_\epsilon(\cdot) = A(\cdot) + \epsilon B(\cdot)$$

where $\epsilon \to 0$.

First order approximation for $\epsilon \mapsto \lambda_j^2(\epsilon)$? Is the parameter dependence holomorphic at $\epsilon = 0$? Good questions.

Spectral perturbation (2)

It is a reasonable assumption that the $A(0)$ and $A(\ell)$ are known. Thus, the correction may be assumed to satisfy $B(0) = B(\ell) = 0$.

After some computations, we get from the perturbation series

$$\lambda_j^2(\epsilon) = \lambda_j^2(0) + \epsilon \cdot \frac{c^2}{2} \int_{[0,\ell]} A \frac{\partial \psi^2_{\lambda_j(0)}}{\partial s} \, d(A^{-1}B) + \mathcal{O}(\epsilon^2).$$

If $B(s) = \delta(s - x)$, then (at least, formally), we get the local sensitivity formula for the eigenvalue λ_j at point $x \in [0, \ell]$:

$$S(x; A, \psi_{\lambda_j}) = \frac{c^2}{2} \left(\frac{\partial^2 \psi^2_{\lambda_j}}{\partial s^2} + \frac{A'}{A} \frac{\partial \psi^2_{\lambda_j}}{\partial s} \right)(x).$$

Points of minimum and maximum sensitivity?

This is akin to Lindstedt–Poincaré perturbation method, 1882 $\rightarrow$.

An application: Local spectral inversion

Suppose you have:

1. A rough idea of an area function of a waveguide, say $A_0(\cdot)$, on $[0, \ell]$, leading to Webster's eigenvalues λ_j, $j \in \mathbb{N}$; and
2. measurement data $\mu_j \approx \lambda_j$ for $j \in J \subset \mathbb{N}$.

Can you improve the approximate area function A_0 based on $\{\mu_j\}$?

The answer is perhaps if the original guess $A_0(\cdot)$ is close enough to $A(\cdot)$ – the true but unknown area function corresponding to $\{\mu_j\}$.

Spectral tuning of Webster's resonator.

(Joint ongoing work with D. Aalto, A. Hannukainen, T. Lukkari.)

Outline of a spectral tuning algorithm (1)

If $|B(s)| \ll A_0(s)$ with $B(0) = B(\ell) = 0$, then there is another way of writing the approximate spectral perturbation:

$$\lambda_j^2(1) \approx \lambda_j^2 - \left\langle B, W_0 \psi_{\lambda_j}^2 \right\rangle_{L^2(0,\ell)}$$

where

$$W_0 := \frac{1}{A_0(s)} \frac{\partial}{\partial s} \left(A_0(s) \frac{\partial}{\partial s} \right).$$

The idea is to find the improved area function estimate $A_1(\cdot) = A_0(\cdot) + B(\cdot)$ where

$$\lambda_j^2 - \mu_j^2 = \left\langle B, W_0 \psi_{\lambda_j}^2 \right\rangle_{L^2(0,\ell)} = \left\langle W_0(A_0^{-1} B), \psi_{\lambda_j}^2 \right\rangle_{L^2(0,\ell)}$$

for all $j \in J$.

Outline of a spectral tuning algorithm (2)

To get a system of linear equations, write

$$A_0(s)^{-1}B(s) = \sum_{k\in K} b_k \phi_{\nu_k}(s)$$

where $c^2 W_0 \phi_{\nu_k} = \nu_k^2 \phi_{\nu_k}$ with boundary conditions convenient for functions $A_0^{-1}B$. Enforce $B(0) = B(\ell) = 0$ by an additional equation if needed.

Solve b_k's from the linear equations, choosing the index sets $J, K \subset \mathbb{N}$ efficiently. Then iterate to get $A_2(\cdot)$ from $A_1(\cdot)$, etc., until convergence ... or a disaster.

Using strictly dissipative boundary conditions leads to non-real λ^2's and the corresponding eigenfunctions. Then you need to do the perturbation analysis separately for $\mathrm{Re}\,\lambda^2$ and $\mathrm{Im}\,\lambda^2$.

Can you hear the area function of a waveguide? (1)

Are there isospectral lossless waveguides with different $A(\cdot)$?

As such, the answer is trivially in positive.

1. You cannot hear the *diameter* of waveguide since $kA(\cdot)$ lead to same Webster's model for all $k > 0$. (However, the acoustic impedance changes with k.)
2. You cannot hear the *direction* of the waveguide (if you have the same boundary conditions at the both ends).

Remember these concerning the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^2$:
M. Kac: "Can you hear the shape of a drum?", 1966.
C. Gordon, D. Webb: "You can't hear the shape of a drum", 1992.

Distictive Regions Model (DRM) in phonetics: Given vowel resonances, can the configuration of the vocal tract be concluded in a test subject?

Can you hear the area function of a waveguide? (2)

Define the operator

$$W^{\#} := A(s)\frac{\partial}{\partial s}\left(\frac{1}{A(s)}\frac{\partial}{\partial s}\right)$$

as the *companion* of the original Webster's operator

$$W := \frac{1}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial}{\partial s}\right).$$

Can you hear the area function of a waveguide? (3)

Are there isospectral lossless waveguides with different $A(\cdot) \in C^\infty[0,\ell]$ satisfying $A(0) = 1$?

Answer is still YES, since the operators W and its companion $W^\#$ are iso(-point-)spectral:

$$\left(\frac{\lambda}{c}\right)^2 \psi_\lambda = \frac{1}{A(s)} \frac{\partial}{\partial s} \left(A(s) \frac{\partial}{\partial s} \right) \psi_\lambda$$

$$\Leftrightarrow$$

$$\left(\frac{\lambda}{c}\right)^2 \cdot \left\{ A(s) \frac{\partial \psi_\lambda}{\partial s} \right\} = A(s) \frac{\partial}{\partial s} \left(\frac{1}{A(s)} \frac{\partial}{\partial s} \right) \cdot \left\{ A(s) \frac{\partial \psi_\lambda}{\partial s} \right\}.$$

Thus, $A(\cdot)$ and $A(\cdot)^{-1}$ define isospectral Webster's resonators (if you play right with the boundary conditions).

N.B! Adding a boundary dissipation term on the waveguide walls changes the conclusion.

Conclusion

"Omne tulit punctum qui miscuit **utile dulci**,
lectorem delectando pariterque monendo."

Horatius, Ars Poetica

"Opera magna"

A. Hannukainen, T. Lukkari, J. Malinen, and P. Palo.
Vowel formants from the wave equation. *J Acoust Soc Am*, 122(1):EL1–EL7, 2007.

D. Aalto, O. Aaltonen, R.-P. Happonen, J. Malinen, P. Palo, R. Parkkola, J. Saunavaara, M. Vainio,
Recording speech sound and articulation in MRI, *Proceedings of BIODEVICES 2011*, 168–173, 2011.

A. Aalto and J. Malinen.
Composition of passive boundary control systems. *Math Control Relat F*, 3(1):1–19, 2013.

T. Lukkari and J. Malinen.
Webster's equation with curvature and dissipation. *arXiv:1204.4075 (submitted), 2013.*

D. Aalto, O. Aaltonen, R.-P. Happonen, P. Jääsaari, A. Kivelä, J. Kuortti, J. M. Luukinen, J. Malinen, T. Murtola, R. Parkkola, J. Saunavaara, and M. Vainio.
Large scale data acquisition of simultaneous MRI and speech. *Appl Acoust*, 83(1):64–75, 2014.

A. Aalto, T. Lukkari, and J. Malinen.
Acoustic wave guides as infinite-dimensional dynamical systems. *ESAIM Contr Optim Ca*, 21(2): 324-247, 2015.

T. Lukkari and J. Malinen.
A posteriori error estimates for Webster's equation in wave propagation. *J Math Anal Appl*, 427(2):941–961, 2015.

A. Aalto, T. Murtola, J. Malinen, D. Aalto, and M. Vainio.
Modal locking between vocal fold and vocal tract oscillations: Simulations in time domain. *arXiv:1506.01395 (submitted), 2015.*

Theses

A. Kivelä.
Acoustics of the vocal tract: MR image segmentation for modelling.
Master's thesis, 2015.

A. Aalto.
Infinite Dimensional Systems: Passivity and Kalman Filter Discretization.
Doctoral dissertation, 2014.

T. Murtola.
Modelling vowel production.
Licentiate thesis, 2014.

P. Palo.
A wave equation model for vowels: Measurements for validation.
Licentiate thesis, 2011.

A. Aalto.
A low-order glottis model with nonturbulent flow and mechanically coupled acoustic load.
Master's thesis, 2009.

Thanks for your patience.
Any questions?

http://speech.math.aalto.fi
https://www.youtube.com/channel/UCDRLICfptS1TQNLkzFjC94g